Analytic vs Smooth Lie groups

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May 24, 2025

There is an equivalence of categories given by the forgetful functor between

$$\big(\text{Real analytic groups, analytic maps}\big) \to \big(\text{Real Lie groups, smooth maps}\big).$$

This map is clearly functorial, as analytic maps are smooth ([Ser92, Part II, Chapter II pg 73] for the faithless), and it is immediate that **the forgetful functor is faithful.**

The definition of smooth Lie group is in any geomtry book, and the definition of analytic groups can be found in [Ser92].

Theorem (Serre, Part II, Chapter V.9, Thm 2 + Remark)**.** *The category of analytic groups over* $\mathbb{R}$ *or* $\mathbb{Q}_p$ *is a full subcategory of of all locally compact topological groups.*

In particular real analytic groups are a full subcategory of real Lie groups, this category is by definition the essential image of the forgetful functor. **Thus the forgetful functor is full.**

Now Whitneys classic result [Whi36] shows that every smooth manifold can be *smoothly* embedded into $\mathbb{R}^n$ as a *analytic* submanifold. First for compact manifolds [Mor58] and then for non-compact manifolds less than a year later [Gra58] showed that every analytic manifold can be *analytically* embedded into R^n. The point is that

Theorem (Morrey-Grauert Theorem)**.** *Every smooth manifold can be made into an analytic manifold that is unique up to analytic isomorphism.*

The proof is that any two embeddings are analytically isomorphic to the original space and therefore to one another.

This is however for manifolds, the final step is that the Baker–Campbell–Hausdorff formula for the Lie group multiplication shows that it is always analytic in a neighborhood of the identity, with the chart being given by the Lie algebra. Thus the manifold can be made analytic and the group structure is already analytic and we have that **the forgetful functor is essentially surjective**. Thus *the forgetful functor is an equivalence.*

References

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